Analysis of Power Electronic Converters Using the Generalized State-Space Averaging Approach

J. Mahdavi, A. Emaadi, M. D. Bellar, and M. Ehsani

***Abstract*— Power electronic converters are periodic time-variant systems, because of their switching operation. The generalized state-space averaging method is a way to model them as time independent systems, defined by a unified set of differential equations, capable of representing circuit waveforms. Therefore, it can be a convenient approach for designing controllers to be applied to switched converters. This brief shows that the generalized state-space averaging method works well only within specific converter topologies and parametric limits, where the model approximation order is not defined by the topology number of components. This point is illustrated with detailed examples from several basic dc/dc converter topologies.**

***Index Terms*— Generalized state-space averaging, power converter modeling.**

I. Introduction

Power electronic converters are mostly periodic variable structure systems, due to their switched operation. The state-space continuous-time modeling of a switched converter, over a switching period T_s with a duty-ratio d, requires two sets of differential equations: one set describes the circuit operation during dT_s time interval, when the switches are on, and the other set to when the switches are off, during $(1-d)T_s$ time interval. During each switch state the circuit behaves as a linear circuit provided the R, L, and C elements are linear. In this case, both sets of state-space differential equations correspond to the exact topological model of the circuit behavior over a switching period T_s. Nevertheless, in order to design regulators for switched converters and to study their stability limits, it is desirable to obtain a unified state-space model description for these converters, valid for the entire switching period T_s. For this purpose, the state-space averaging method was proposed in [1], and it has been used in many power converters such as [2] and [3]. Reference [4] showed the effectiveness and simplicity of this method for PWM dc/dc converter design. This method has been proved to be very useful when the circuit state variables have small variations around the operating point. This condition means that, during an arbitrary period of time, the dc term should be the dominant component when a Fourier series expansion is applied to a circuit state variable waveform. Consequently, this approach is not suitable for modeling converters which have dominant oscillatory behavior, such as the resonant type converters or large ripple PWM converters. The generalized state-space averaging method [5] is defined to provide a more general state-space average model, capable of representing circuit state variable waveforms without discontinuities. In this case, the circuit state variables, i.e., capacitor voltages and inductor currents, meet the constraint imposed by continuity property that provides bounded capacitor currents and inductor voltages [6] and [7]. From the topological point of view, it means that there are no degenerate all C and/or degenerate all L cut-sets in the converter circuit graph [8]. Therefore, analysis of converters with ideal switches and parasitic components (capacitors or inductors) forming loops must be considered with more care. With the generalized state-space averaging method, the circuit state variables are approximated by a Fourier series expansion with time-dependent coefficients. This representation results in an unified time-invariant set of differential equations where the state variables are the coefficients of the corresponding Fourier series of the circuit state variables. Thus, the greater the order of harmonics described in the model, the closer the results will be to the exact topological state-space solution. In practice, some simplifying assumptions can be considered in order to reduce the number of Fourier terms, and hence, to simplify the calculations. The generalized state-space averaging method has already been used in the modeling of some switched power converters [9], however a better understanding on this method is still required. In this paper, the generalized state-space averaging model is applied to the basic dc/dc topologies, such as the Buck, Boost, Buck-Boost, and Cuk. Simulation results are obtained and compared to the exact topological state-space model. It is assumed that all the converters shown here are operating in continuous conduction mode with switching frequency T_s and duty-ratio d. It can be seen that some parameter variations, such as the usual duty-ratio control, will influence the performance of these models. State-space averaging model results are also shown. Comparison of above mentioned averaging methods is also presented.

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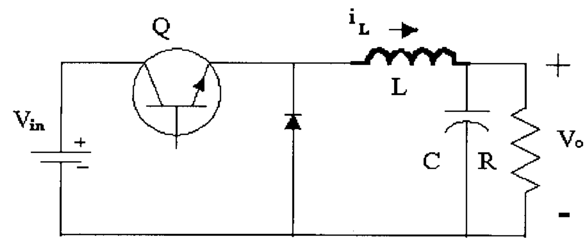


Fig. 1. Buck converter ($V_{\text{in}} = 20$ V, $R = 10\ \Omega$, $L = 1$ mH, $C = 10\ \mu$F, $T_s = 0.1$ ms).

II. Analysis of Basic DC/DC Converters

A. Buck Converter

Fig. 1 shows the Buck converter. To apply the generalized state-space averaging method, first a commutation function $u(t)$ is defined in (1). This function depends on the circuit switching control, which determines when the circuit topology changes according to time. The unified set of circuit state variable equations (2) is obtained by applying (1) to the two sets of topological circuit state-space equations.

$$u(t) = \begin{cases} 1, & 0 < t < dT_s \\ 0, & dT_s < t < T_s \end{cases} \tag{1}$$

$$\frac{di_L}{dt} = \frac{1}{L}(V_{\text{in}} u(t) - v_o)$$
$$\frac{dv_o}{dt} = \frac{1}{C}\left(i_L - \frac{v_o}{R}\right). \tag{2}$$

Nevertheless, in the set of equations of the generalized state-space average model, the actual state-space variables are the Fourier coefficients of the circuit state variables which are, in this case, i_L and v_o. Using the first-order approximation to obtain i_L and v_o, one has

$$i_L = \langle i_L \rangle_{-1} e^{-j\omega t} + \langle i_L \rangle_0 + \langle i_L \rangle_1 e^{j\omega t} \tag{3}$$

$$v_o = \langle v_o \rangle_{-1} e^{-j\omega t} + \langle v_o \rangle_0 + \langle v_o \rangle_1 e^{j\omega t} \tag{4}$$

where ω is the fundamental frequency and

$$\begin{aligned} \langle i_L \rangle_1 &= x_1 + jx_2, \langle i_L \rangle_0 = x_5 \\ \langle v_o \rangle_1 &= x_3 + jx_4, \langle v_o \rangle_0 = x_6. \end{aligned} \tag{5}$$

Since i_L and v_o are real,

$$\langle i_L \rangle_{-1} = \langle i_L \rangle_1^*, \quad \langle v_o \rangle_{-1} = \langle v_o \rangle_1^* \tag{6}$$

where the operator * means the conjugate of a complex number. By applying the time derivative property of the Fourier coefficients in (2), and further substituting the Fourier coefficients of the commutation function $u(t)$

$$\langle u \rangle_0 = d, \quad \langle u \rangle_1 = \frac{\sin\,2\pi d + j(\cos\,2\pi d - 1)}{2\pi} \tag{7}$$

one comes to

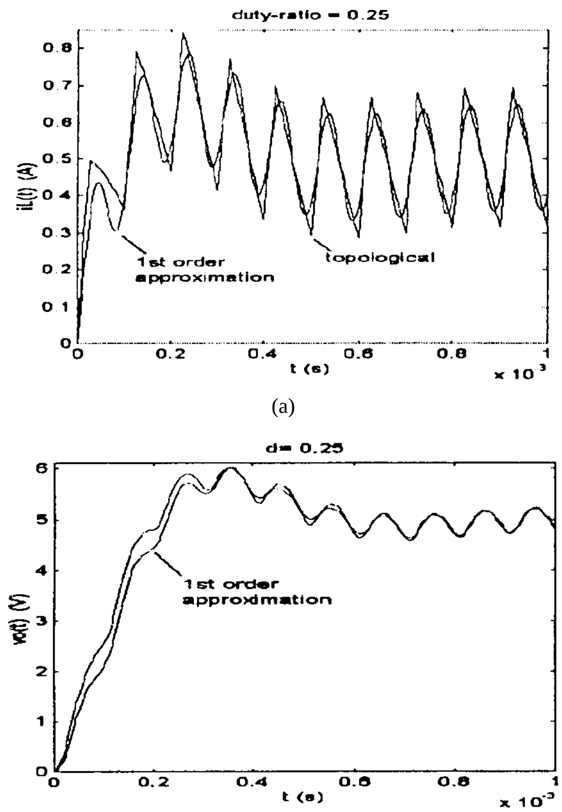
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \\ \dot{x}_6 \end{bmatrix} = \begin{bmatrix} 0 & \omega & \frac{-1}{L} & 0 & 0 & 0 \\ -\omega & 0 & 0 & \frac{-1}{L} & 0 & 0 \\ \frac{1}{C} & 0 & \frac{-1}{RC} & \omega & 0 & 0 \\ 0 & \frac{1}{C} & -\omega & \frac{-1}{RC} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{-1}{L} \\ 0 & 0 & 0 & 0 & \frac{1}{C} & \frac{-1}{RC} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} + \begin{bmatrix} \frac{\sin\,2\pi d}{2\pi L} V_{\text{in}} \\ \frac{-\sin^2\,\pi d}{\pi L} V_{\text{in}} \\ 0 \\ 0 \\ \frac{d}{L} V_{\text{in}} \\ 0 \end{bmatrix} \tag{8}$$

where V_{in} is a constant. Equation (8) is the generalized state-space averaged model of the Buck converter (Fig. 1). The circuit state variables are calculated and given by

$$i_L = x_5 + 2x_1\,\cos\,\omega t - 2x_2\,\sin\,\omega t \tag{9}$$

$$v_o = x_6 + 2x_3\,\cos\,\omega t - 2x_4\,\sin\,\omega t. \tag{10}$$

Fig. 2 shows $i_L(t)$ and $v_o(t)$ when the duty-ratio d is 0.25, using the exact topological model and the generalized state-space averaging model. In Fig. 2(a) and (b) the first-order approximation is used, as given by (9) and (10). If the second-order approximation is used, the corresponding model matrix will have 10 real state variables. Indeed, the use of higher order approximations improves the model accuracy, though at the expense of more complexity in the calculations. Fig. 3 shows the exact topological model together with the first-order and zero-order approximation results for $i_L(t)$ and $v_o(t)$ when the duty-ratio d is 0.5 [Fig. 3(a) and (b)]. The zero-order approximations correspond to the Fourier series dc components, x_5 and x_6, of the circuit state variables i_L and v_o, respectively. The components x_5 and x_6 are also referred as the corresponding moving averages of the current $i_L(t)$ and the voltage $v_o(t)$. In this converter the simulation results show that there is no difference between the moving averages provided by the zero-order approximation and by the state-space



(b)

Fig. 2. Simulation of (a) $i_L(t)$ and (b) $v_o(t)$, when d = 0.25, by the exact topological model and first-order approximation.

averaging model. This fact is explained as follows. From (8) the zero-order terms are described as

$$\dot{x}_5 = -\frac{1}{L}x_6 + \frac{d}{L}V_{\text{in}}, \quad \dot{x}_6 = \frac{1}{C}x_5 - \frac{1}{RC}x_6. \tag{11}$$

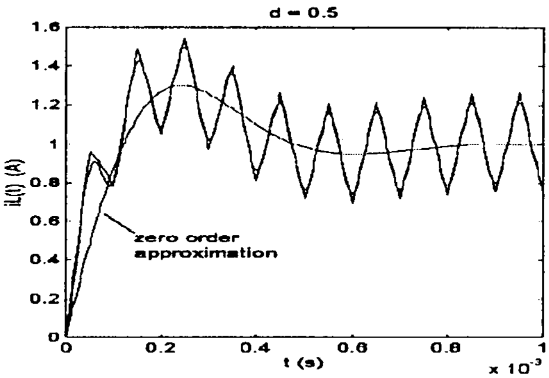
Whereas for the case of the state-space averaging model, it can be shown that

$$\dot{I}_L = -\frac{1}{L}V_o + \frac{d}{L}V_{\text{in}}, \quad \dot{V}_o = \frac{1}{C}I_L - \frac{1}{RC}V_o \tag{12}$$

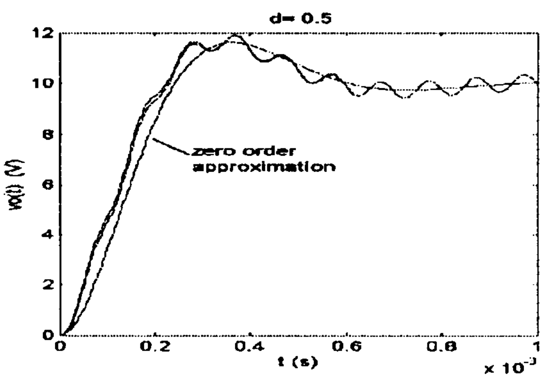
where I_L and V_o are the moving average values of $i_L(t)$ and $v_o(t)$, respectively. It can be seen that the sets (11) and (12) are identical. The steady-state average values are also the same, and they depend only on the duty-ratio [10]. Nevertheless, this does not happen for other converters, as it will be shown later in the case of the buck-boost converter. By looking at Figs. 2 and 3, one can see that the first-order approximation is better when d = 0.5. This fact will be explained in the next sections.

B. Boost Converter

Fig. 4 shows the Boost converter. Using first-order approximation, and the same procedure as applied to the Buck converter, $i_L(t)$ and $v_o(t)$ can be calculated, and these results are shown in Fig. 5. In this figure, the topological and the first-order approximation models are presented for the case when d = 0.25. Here also, it can be verified that the best approximation occurs when d = 0.5. It can be seen that the voltage $v_o(t)$, as given by the topological model, presents more harmonic content (sharp peaks) than in Figs. 2 and 3. Therefore, the first-order approximation in the previous $v_o(t)$ (Figs. 2 and 3) was better. In this case, the generalized state-space averaged model equations have more coupling terms between the state variables than in (8), and this accounts for more complexity in the system equations.



(a)



(b)

Fig. 3. Simulations by using the exact topological model, zero-order and first-order approximations when (a) and (b) d = 0.5.

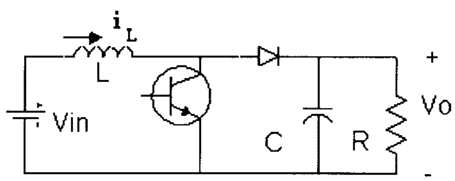


Fig. 4. Boost converter ($V_{\rm in}$ = 20 V, R = 10 Ω, L = 1 mH, C = 10 μF, T_s = 0.1 ms).

C. Buck-Boost Converter

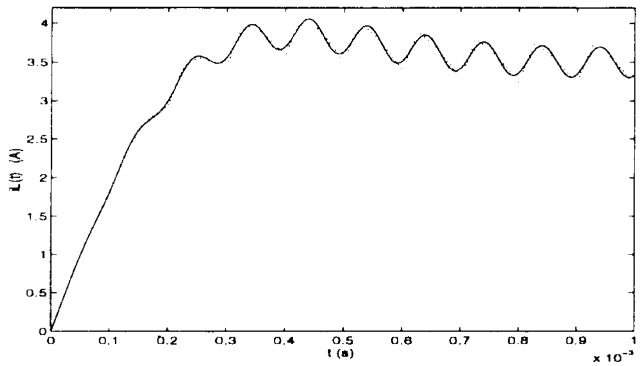
The Buck-Boost converter is shown in Fig. 6. In Fig. 7, the simulated waveforms of $i_L(t)$ and $-v_o(t)$ are presented. Fig. 7(a) and 7(b) show the results of the topological and first-order approximation models when d = 0.25. In Fig. 7(c) and (d), the topological model, the first-order, zero-order and the state-space averaging models are presented when d = 0.5. It can be seen that there is a difference between the zero-order and the state-space averaging approximations. In fact, the Fourier zero-order expansion gives a more precise moving average of a state variable. In fact, it can be shown that the set of generalized state-space averaging equations gives

$$\dot{x}_5 = -\frac{\sin\,2\pi d}{\pi L}x_3 + \frac{2\,\sin^2\,\pi d}{\pi L}x_4 + \frac{1-d}{L}x_6 + \frac{d}{L}V_{\rm in} \tag{13}$$

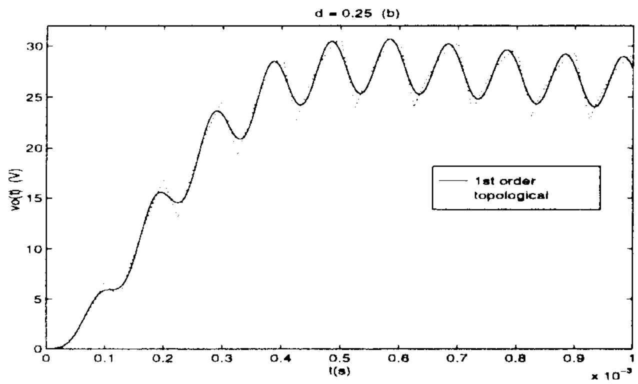
where x_5 and x_6 are the moving average of $i_L(t)$ and $v_o(t)$, respectively. For the case of the state-space averaging model, the moving average V_o of the voltage $v_o(t)$ is obtained as

$$\dot{I}_L = \frac{1-d}{L}V_o + \frac{d}{L}V_{\rm in}. \tag{14}$$

Hence, (13) and (14) show that both methods provide different moving averages, as well as different steady-state average values. However, V_o is approximately equal to x_6 when the terms x_3 and x_4 (first-order terms) are negligible. This happens when the switching frequency $1/T_s$ is sufficiently high [10]. As in the previous converters, it can also be seen here that, when d = 0.5 the model results show better approximation. Moreover, the current $i_L(t)$ shows more sharp peaks than the voltage $v_o(t)$. Consequently, the first-order approximation is better for the voltage than for the current. This fact can be explained as follows. For a properly designed converter operating in continuous conduction mode, it is necessary that the switching period be much smaller than the circuit time constants (low-pass filtering condition). This results in a triangular wave shape for the inductor current and a piecewise quadratic wave shape for the capacitor ripple voltage. If d = 0.5 the Fourier series of periodic signals contains only odd harmonics. For the triangular function the third harmonic is 1/9th the amplitude of the fundamental while for the quadratic function this ratio is 1/27. Therefore, when d = 0.5 a better approximation for the voltage state variable is obtained [7].



(a)



(b)

Fig. 5. Boost Converter simulation: topological and first-order approximation models.

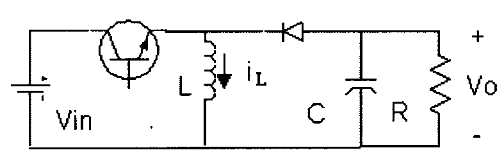


Fig. 6. Buck-boost converter ($V_{\rm in}$ = 20 V, R = 10 Ω, L = 1 mH, C = 10 μF, T_s = 0.1 ms).

D. Cuk Converter

In the case of the Cuk converter (Fig. 8), the first-order approximation produces twelve real state variables. In Fig. 9 the simulated output voltage $-v_o(t)$ waveforms, by the first-order and exact topological models, are presented for d = 0.25. The results from both models are coincident. Though the Cuk converter is apparently more

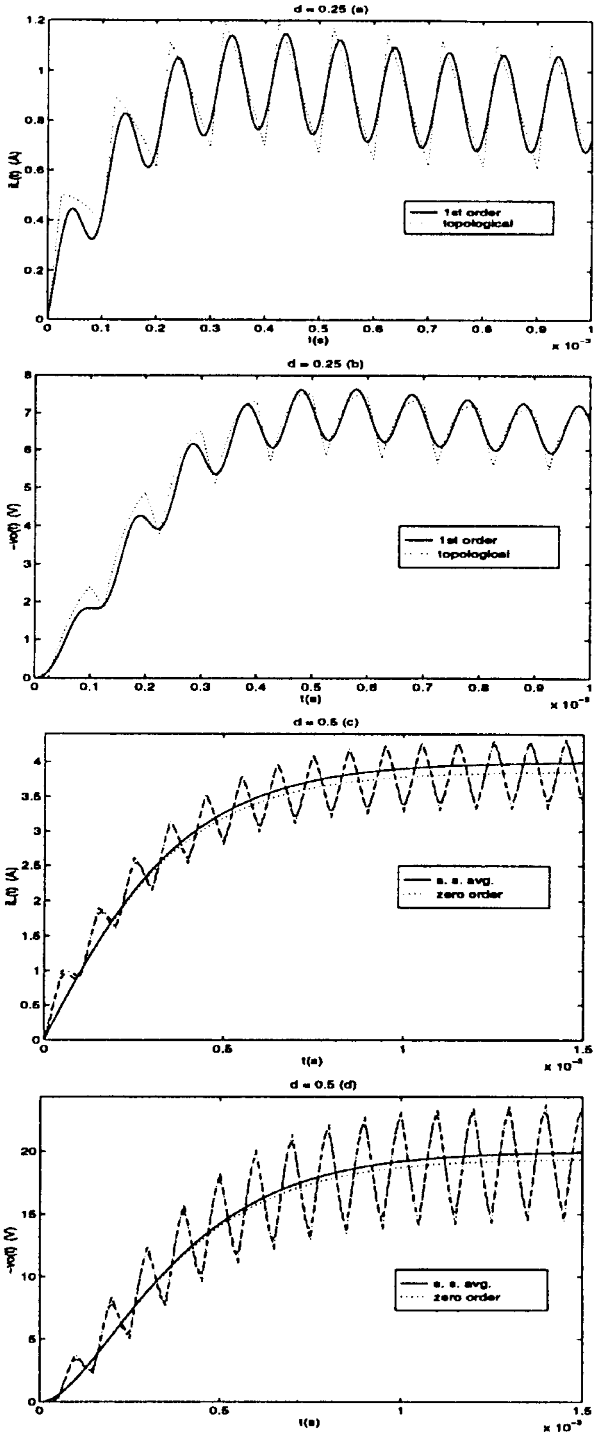


Fig. 7. Buck-boost converter simulation results by the (a), (b) exact topological and first-order approximately, and (c), (d) the state-space averaged, zero-order, exact topological and first-order modeling methods.

complex than the previous ones, in terms of number of components, the first-order approximation can suit well provided the low-pass filtering condition, is met.

III. Conclusion

In this brief, the generalized state-space averaging method was applied to the basic dc/dc single-ended topologies. Simulation results were compared to the exact topological state-space model and to the well-known state-space averaging method. It becomes evident that when the switching frequency is not much higher than the converter natural frequencies, the approximation order is an important factor in improving the model accuracy, though at the expense of increasing the calculations. A detailed analysis is shown regarding the influence of the switching frequency on the moving average obtained from the generalized state-space averaging and the state-space averaging methods. It can be observed that, the results of the generalized state-space averaged model with first-order approximation are closer approximations of the corresponding ones of the topological model, when the duty-ratio is around 0.5 (absence of even harmonics) and for specific state variables. It can also be seen that, the topology complexity, in terms of number of components, does not determine the approximation order for a satisfactory model. Since this method can provide a unified time-invariant state-space model of the converters, a software like Matlab can be used for controller design and stability analysis.

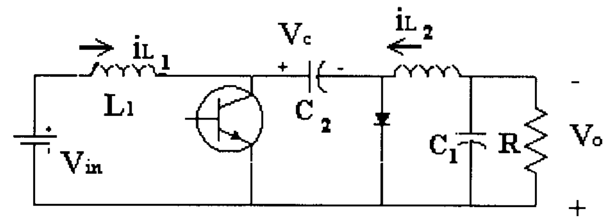


Fig. 8. Cuk converter.

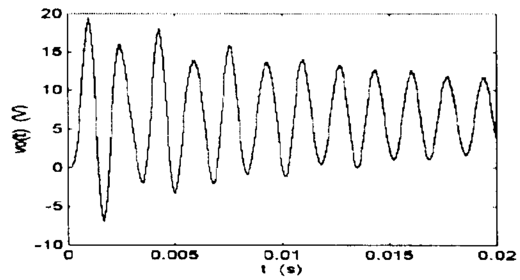


Fig. 9. Cuk converter ($V_{\rm in}$ = 20 V, R = 10 Ω, $C1$ = 220 μF, $C2$ = 200 μF, $L1$ = 180 μH, $L2$ = 150 μH, T_s = 0.1 ms). Simulated output voltage $-v_o(t)$ when d = 0.25.

References

[1] R. D. Middlebrook and S. Cuk, "A general unified approach to modeling switching power converter stages," in *IEEE PESC Rec.*, 1976, pp. 18–34.

[2] M. O. Bilgic and M. Ehsani, "Analysis of single flying capacitor converter by the state-space averaging technique," in *Proc. 1988 IEEE Int. Symp. Circuits Syst. (ISCAS'88)*, Finland, pp. 1151–1154.

[3] ——, "Analysis of inductor-converter bridge by means of state-space averaging technique," in *Proc. 1988 IEEE Power Electron. Specialists Conf. (PESC'88)*, pp. 116–121.

[4] N. Mohan, T. M. Undeland, and W. P. Robbins, *Power Electronics—Converters, Applications and Design*, second ed. New York: Wiley, 1995.

[5] S. R. Sanders, J. M. Noworolski, X. Z. Liu, and G. C. Verghese, "Generalized averaging method for power conversion circuits," *IEEE Trans. Power Electron.*, vol. 6, pp. 251–259, Apr. 1991.

[6] J. G. Kassakian, M. Schiecht, G. Vergese, *Principles on Power Electronics*. Reading, MA: Addison-Wesley, 1991.

[7] A. V. Oppenheim, A. S. Willsky, and I. T. Young, *Signal and Systems*. Englewood Cliffs, NJ: Prentice-Hall, 1983.

[8] L. O. Chua, C. A. Desoer, and E. S.Kuh, *Linear and Nonlinear Circuits*. New York: McGraw-Hill, 1987.

[9] A. W. Green, "Modeling a push-pull parallel resonant converter using generalized state-space averaging," *Inst. Elect. Eng. Proc. B*, vol. 140, pp. 350–356, Nov. 1993.

[10] J. Xu and J. Yu, "An extension of time averaging equivalent circuit analysis for DC-DC converters," in *Proc. IEEE ISCAS'89*, pp. 2060–2063.